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Tame prinjective type and Tits form of two-peak posets I [1]

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Abstract

The main aim of this paper is to give a simple criterion for a finite poset I with two maximal elements to have the category I-spr of socle projective representations of tame representation type. Our main result is Theorem 1 which asserts that for any upper chain reducible poset I with two maximal elements (see Definition 8) the category I-spr is of tame representation type if and only if the Tits quadratic form $q_I : \mathbb{Q}^I \to \mathbb{Q}$ (1.1) of I is weakly non-negative, or equivalently, if and only if I does not contain as a peak subposet any of the one-peak posets $\mathcal{N}_1^*, \ldots, \mathcal{N}_6^*$ of Nazarova presented in Theorem 1 or any of the 41 two-peak posets listed in Table 1.

1. Introduction

Representations of partially ordered sets (posets) over a field have long been recognized to be of importance for the study of indecomposable representations of groups and finite dimensional algebras, lattices over orders, abelian groups and Cohen–Macaulay modules (see [1,3,10,12,16]). They provide elegant, simple and powerful tools for studying the representation type of algebras and for determining their indecomposable representations. Many of the well-known classification results and representation type criteria depend essentially on the theory of poset representations and related ideas, because many of the representation theory problems can be reduced to the corresponding problems for I-spaces over a finite poset I, and there are simple criteria of Kleiner, of

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Nazarova (see [12,16]) and of the second author [18] for determining the representation type of the category I-spr of peak I-spaces (see [1,10,12,16]).

It was shown in [14-18] that the study of a class of non-Schurian vector space categories and the study of lattices over a class of orders reduce to the study of socle projective representations of finite posets I with distinguished maximal elements (called *peaks* of I). This leads to the notion of a peak I-space introduced in [15,18] and to the natural representation type question for the categories I-spr of peak I-spaces. Posets I for which the category I-spr is of finite representation type are characterized in [18].

Throughout this paper K is an algebraically closed field and we denote by $(I, \preceq)$ a finite *poset* (i.e. partially ordered set) with respect to the partial order $\preceq$. We shall write $i \prec j$ if $i \preceq j$ and $i \neq j$. For the sake of simplicity we write I instead of $(I, \preceq)$. We denote by $\max I$ the set of all maximal elements of I and I will be called an *r-peak poset* if $|\max I| = r$.

We recall from [18] that the category I-spr of *peak I-spaces* (or *socle projective representations* of I) over the field K is defined as follows. The objects of I-spr are systems

$$\mathbf{M} = (M_j)_{j \in I}$$

of finite-dimensional K-vector spaces M_j such that $M_j \subseteq M^\bullet = \bigoplus_{p \in \max I} M_p$ for all $j \in I$, $\pi_p(M_j) = 0$ for $j \npreceq p \in \max I$ and $\pi_j(M_i) \subseteq M_j$ for $i \prec j \in I$, where π_j is the composed map

$$M^\bullet \xrightarrow{\pi_j'} \bigoplus_{j \preceq p \in \max I} M_p \hookrightarrow M^\bullet$$

and π_j' is the direct summand projection.

By a map $f : \mathbf{M} \to \mathbf{M}'$ in I-spr we mean a system $f = (f_p)_{p \in \max I}$ of K-linear maps $f_p : M_p \to M_p'$, $p \in \max I$, such that $(\bigoplus_{p \in \max I} f_p)(M_j) \subseteq M_j'$ for all $j \in I$. It is clear that I-spr is an additive category with the finite unique decomposition property. Moreover, I-spr is closed under taking kernels and according to [2] the category I-spr has Auslander-Reiten sequences, source maps and sink maps (see [12]).

If I has a unique maximal element $*$, then the notion of a peak I-space coincides with the notion of a $(I \setminus \{*\})$-space in the sense of Gabriel [5].

Following [18] we associate to any two-peak poset I with $\max I = \{*, +\}$ the rational *Tits quadratic form* $q_I : \mathbb{Q}^I \to \mathbb{Q}$,

$$q_I(z) = \sum_{j \in I} z_j^2 + \sum_{i \prec j,\, j \notin \max I} z_i z_j - \sum_{i \prec *} z_i z_* - \sum_{j \prec +} z_j z_+, \tag{1.1}$$

where $z = (z_j)_{j \in I}$ is a vector in $\mathbb{Q}^I$ viewed as a function $z : I \to \mathbb{Q}$, $j \mapsto z_j$.

The aim of this paper is to study two-peak posets I for which the category I-spr is of tame representation type (see [8], [18] and Section 2) by means of the Tits form q_I.

We recall from [8] and [11] that the Tits form $q_I : \mathbb{Q}^I \longrightarrow \mathbb{Q}$ (1.1) of any such poset I is weakly non-negative, that is, $q_I(z) \geq 0$ for any vector $z \in \mathbb{N}^I$. Posets with weakly non-negative Tits form are characterized in [8] and [9] (see Theorem 5). Therefore Theorem 5 gives necessary conditions for I to have the category I-spr of tame representation type.

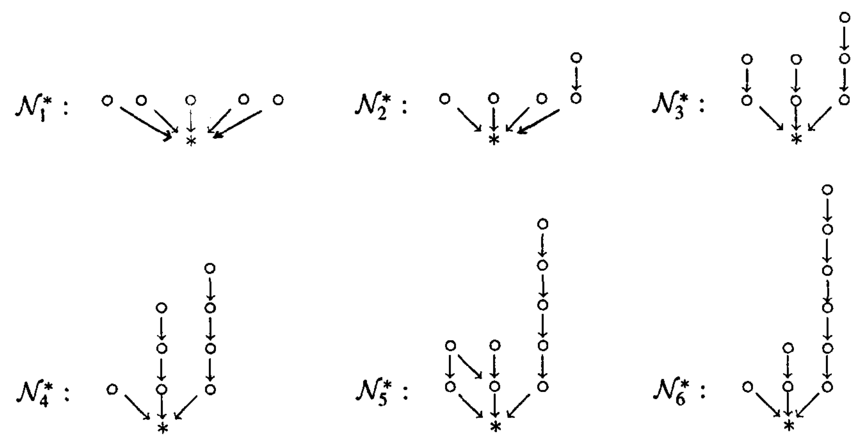
The main aim of this paper is to give a simple criterion in terms of the Tits quadratic form for a finite poset I with two maximal elements to have the category I-spr of socle projective representations of tame representation type. Our main result is the following theorem proved in Section 5.

Theorem 1. *Suppose that K is an algebraically closed field and I is a finite poset having exactly two maximal elements $\star$ and $+$. Moreover, we suppose that I is upper chain reducible in the sense of Definition* 8. *Then the following conditions are equivalent.*

(a) *The category I-spr is of tame representation type.*

(b) *The Tits quadratic form $q_I : \mathbb{Q}^I \to \mathbb{Q}$* (1.1) *is weakly non-negative.*

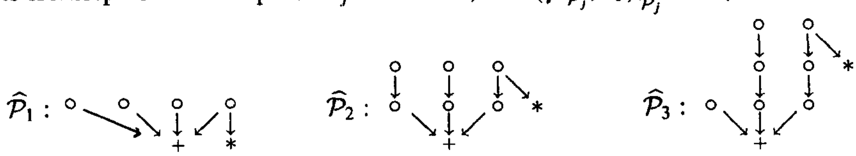
(c) *The poset I does not contain as a peak subposet a poset isomorphic to any of the one-peak enlargements*

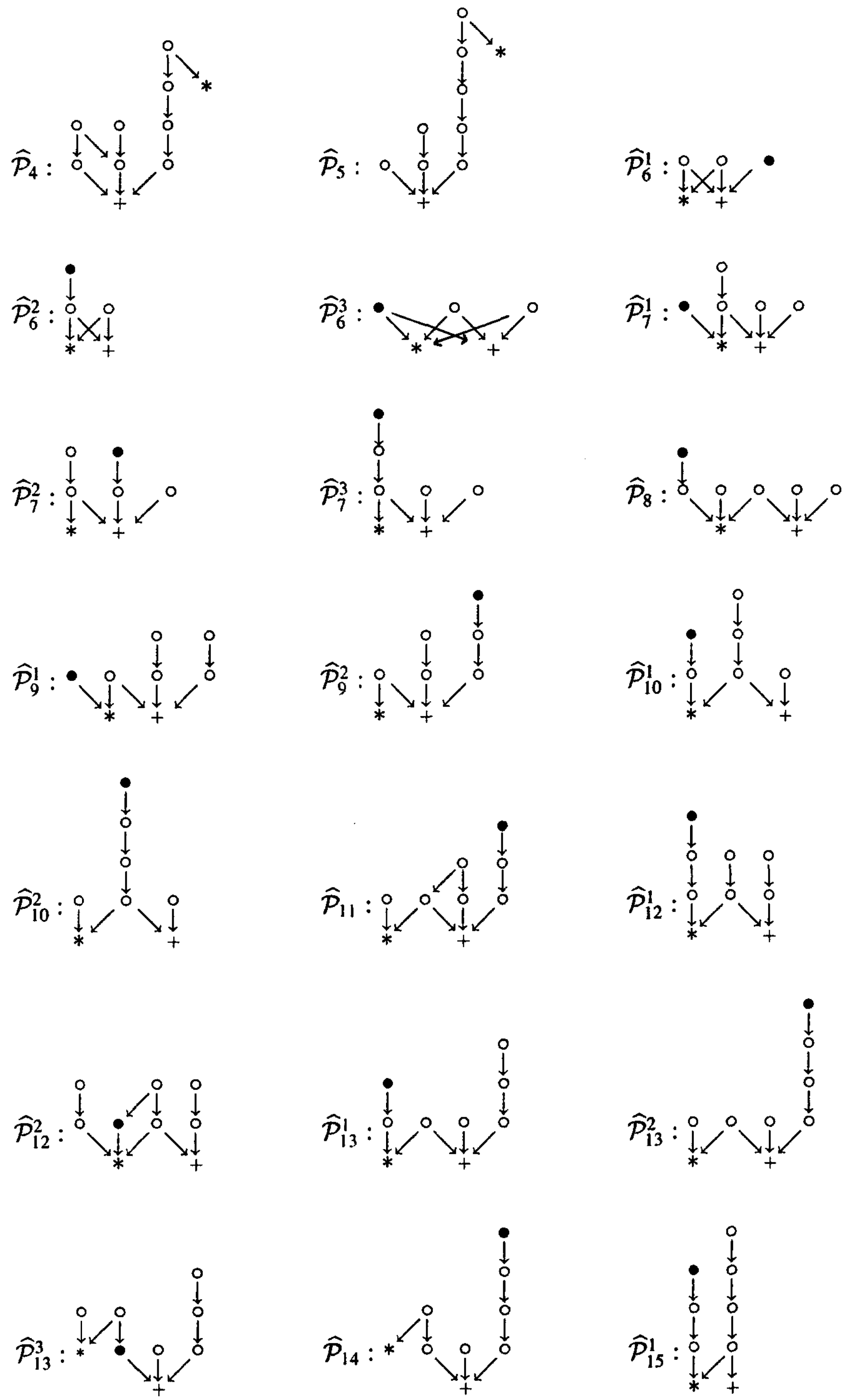


of the Nazarova's hypercritical posets $\mathcal{N}_1 = (1,1,1,1,1)$, $\mathcal{N}_2 = (1,1,1,2)$, $\mathcal{N}_3 = (2,2,3)$, $\mathcal{N}_4 = (1,3,4)$, $\mathcal{N}_5 = (N,5)$, $\mathcal{N}_6 = (1,2,6)$ (see [16]) *or any of the* 41 *posets listed in Table* 1 *below.*

A subposet I' of I is said to be a *peak subposet* if $I' \cap \max I = \max I'$.

Table 1. *Minimal two-peak posets of wild prinjective type.* In the following tables the solid vertex $\bullet$ marked in the diagram $\widehat{\mathcal{P}}^i_j$ denotes a vertex such that the poset $\widehat{\mathcal{P}}^i_j \setminus \{\bullet\}$ is isomorphic with the poset $\mathcal{P}_j$ in Table 2, and $(\mu_{\mathcal{P}_j}, \varepsilon_\bullet)_{\widehat{\mathcal{P}}_j} < 0$ (see Theorem 5(b)).




$\widehat{\mathcal{P}}_4$:
$\widehat{\mathcal{P}}_5$:
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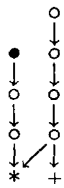
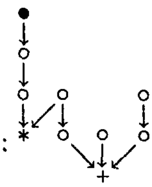
$\widehat{\mathcal{P}}_{14}$:
$\widehat{\mathcal{P}}_{15}^1$:

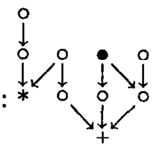
$\widehat{\mathcal{P}}^2_{15}$:



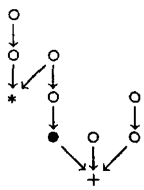
$\widehat{\mathcal{P}}^1_{16}$:



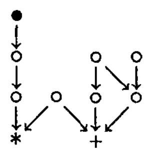
$\widehat{\mathcal{P}}^2_{16}$:



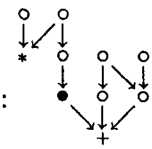
$\widehat{\mathcal{P}}^3_{16}$:



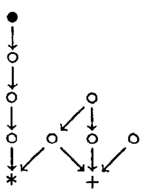
$\widehat{\mathcal{P}}_{17}$:



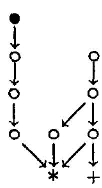
$\widehat{\mathcal{P}}_{18}$:



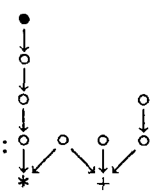
$\widehat{\mathcal{P}}_{19}$:



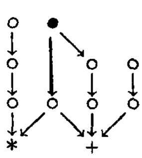
$\widehat{\mathcal{P}}_{23}$:



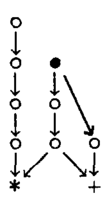
$\widehat{\mathcal{P}}^1_{24}$:



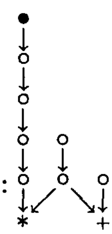
$\widehat{\mathcal{P}}^2_{24}$:



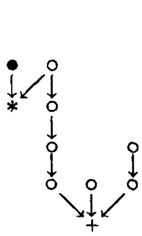
$\widehat{\mathcal{P}}^1_{25}$:



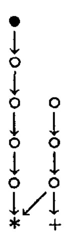
$\widehat{\mathcal{P}}^2_{25}$:



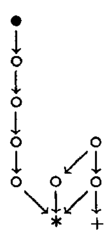
$\widehat{\mathcal{P}}_{26}$:

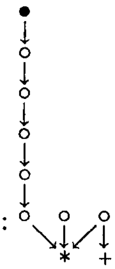


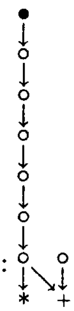
$\widehat{\mathcal{P}}_{27}$:



$\widehat{\mathcal{P}}_{29}$:



$\widehat{\mathcal{P}}_{30}$: 

$\widehat{\mathcal{P}}_{31}$: 

Theorem 1 is proved in Section 5 by associating to any upper chain reducible two-peak poset I the one-peak poset $\xi_C I$ described in Corollary 9, the $\mathbb{Q}$-linear map $\widetilde{\xi}$: $\mathbb{Q}^{\xi_C I} \to \mathbb{Q}^I$ (4.5) and the defect quadratic form $q_I^- = q_{\xi_C I} - q_I \circ \widetilde{\xi} : \mathbb{Q}^{\xi_C I} \to \mathbb{Q}^I$ (4.6). The main tool for the proof of Theorem 1 is Lemma 12, which reduces the problem from two-peak posets to one-peak posets, and allows us to apply Nazarova's characterization of one-peak posets of tame representation type (see [16, Theorem 15.3]).

Our Theorem 1 is a counterpart for two-peak posets of the Nazarova's characterization of one-peak posets of tame representation type. It has many useful applications, because it follows from [13], [17, Section 6] and [18, Section 4] that the results of this paper can be applied to the study of orders of tame lattice type and to the study of non-Schurian vector space categories of tame representation type (see [17, 6.2–6.6]). Our Theorem 1 is essentially applied in [20].

Throughout we denote by $\mathrm{mod}(R)$ the category of finitely generated right modules over a ring R. Given X in $\mathrm{mod}(R)$ and an integer $t \geq 0$ we denote by X^t the direct sum of t copies of X. For any $j \in I$ we set

$$j^{\nabla} = \{i \in I \mid i \preceq j\}. \tag{1.2}$$

2. Preliminaries and notation

Throughout this paper we fix a finite poset I and we suppose that

$$I = \{1, \ldots, n, *, +\}, \qquad \max I = \{*, +\}.$$

Moreover, we suppose that the order relation $\prec$ in I is such that $i \prec j$ implies that $i < j$ in the natural order. We can always achieve this by a suitable renumbering of the elements in I. We view the poset I as a quiver with the commutativity relation induced by the ordering $\prec$, and we denote by KI the path algebra of I with coefficients in the field K. Given j in I we denote by e_j the standard primitive idempotent corresponding to j. It follows from our assumption that KI has the following upper triangular matrix form

$$KI = \begin{bmatrix} K & K_{12} & \dots & K_{1n} & K_{1*} & K_{1+} \\ 0 & K & \dots & K_{2n} & K_{2*} & K_{2+} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & K & K_{n*} & K_{n+} \\ 0 & 0 & \dots & 0 & K & 0 \\ 0 & 0 & \dots & 0 & 0 & K \end{bmatrix} \tag{2.1}$$

where $K_{ij} = K$ if $i \prec j$ and $K_{ij} = 0$ otherwise. A right ideal Y in KI is simple if and only if Y is isomorphic to one of the ideals e_*KI and e_+KI (called *peak ideals* of KI). The algebra KI is of finite global dimension [18, Lemma 2.1].

The finite-dimensional right KI-modules will be identified with the systems

$$X = (X_i; {}_jh_i)_{i,j \in I},$$

where X_i is a finite-dimensional K-vector space and ${}_jh_i : X_i \longrightarrow X_j$, $i \prec j$, are K-linear maps such that ${}_th_j \cdot {}_jh_i = {}_th_i$ for all $i \prec j \prec t$ in I.

Following [11] we call the right KI-module X *prinjective* if X is finitely generated and the right module $\mathrm{res}_{I^-}(X) = Xe^-$ over the algebra

$$KI^- \cong e^-(KI)e^- \cong KI/\mathrm{soc}(KI_{KI})$$

is projective, where

$$I^- = I \setminus \max I \quad \text{and} \quad e^- = \sum_{j \in I^-} e_j.$$

It is easy to prove that a module X in $\mathrm{mod}(KI)$ is prinjective if and only if there exists an exact sequence

$$0 \longrightarrow (e_*KI)^{s_*} \oplus (e_+KI)^{s_+} \longrightarrow P \longrightarrow X \longrightarrow 0$$

in $\mathrm{mod}(KI)$, where P is a projective KI-module and $s_*, s_+ \geq 0$.

A module X in $\mathrm{mod}(KI)$ is said to be socle projective if the socle $\mathrm{soc}\, X$ of X is projective and isomorphic to a direct sum of copies of e_*KI and e_+KI (see [16]).

We denote by $\mathrm{prin}(KI)$ and by $\mathrm{mod}_{\mathrm{sp}}(KI)$ the categories of finitely generated prinjective right KI-modules and socle projective right KI-modules respectively. It follows from [11] that the categories $\mathrm{prin}(KI)$ and $\mathrm{mod}_{\mathrm{sp}}(KI)$ have Auslander–Reiten sequences, source maps and sink maps. An interpretation of prinjective KI-modules in terms of partitioned matrices is given in [15,18].

We recall from [18] that there exist two additive functors

$$\mathrm{mod}_{\mathrm{sp}}(KI) \xrightarrow[\sim]{\Theta'_I} I\text{-spr} \xleftarrow{\Theta_I} \mathrm{prin}(KI). \tag{2.2}$$

The functor Θ'_I is an equivalence of categories, whereas the functor Θ_I is full, dense and $\mathrm{Ker}\,\Theta_I = [e_jKI^-; j \in I^-]$, i.e. $\mathrm{Ker}\,\Theta_I$ consists of all maps in $\mathrm{prin}(KI)$ having a factorization through a direct sum of copies of the projective KI^--modules e_jKI^-, $j \in I^-$. In particular the functor Θ_I establishes a one to one correspondence between the isomorphism classes of indecomposable modules in $\operatorname{prin}(KI)$ non-isomorphic to e_iKI^-, $i \in I^-$, and the isomorphism classes of indecomposable I-spaces. Here we view e_iKI^- as a right KI-module via the natural algebra surjection $KI \to KI^-$ (see (2.1)).

Throughout this paper the equivalence Θ'_I will be frequently used as an identification.

Following [18, (2.7)] given a KI-module X we define its *coordinate vector* $\mathbf{cdn}(X) \in \mathbb{Z}^I$ by the formula

$$(\mathbf{cdn}(X))(i) = \begin{cases} \dim_K \operatorname{top}(X)e_i & \text{for } i \in I^-, \\ \dim_K Xe_i & \text{for } i \in \max I. \end{cases} \tag{2.3}$$

We recall from [8] and [18]) that the category $\operatorname{prin}(KI)$ is said to be of *tame representation type* (resp. I-spr is of tame representation type) if it is not of finite representation type and for every $m \geq 1$ there exist functors

$$\widehat{N}^{(1)}, \dots, \widehat{N}^{(r)} : \operatorname{ind}_1(\operatorname{mod}(K[y])) \longrightarrow \operatorname{mod}(KI)$$

forming an *almost parameterizing family* for the full subcategory $\operatorname{ind}_m(\operatorname{prin} KI)$ (resp. for $\operatorname{ind}_m(I\text{-spr})$) of $\operatorname{mod}(KI)$ consisting of representatives of the isomorphism classes of indecomposable objects in $\operatorname{prin}(KI)$ (resp. in I-spr) of K-dimension m (see [16, Chapters 14 and 15]). This means that $\widehat{N}^{(j)} = (-) \otimes_{K[t]} N^{(j)}$ and $N^{(j)}$ is a $K[t]$-KI-bimodule which is finitely generated as a $K[y]$-module for $j = 1, \dots, r$ and $\operatorname{Im} \widehat{N}^{(1)} \cup \cdots \cup \operatorname{Im} \widehat{N}^{(r)}$ contains up to isomorphism all but finitely many objects in $\operatorname{ind}_m(\operatorname{prin}(KI))$ (resp. in $\operatorname{ind}_m(I\text{-spr})$).

Let $\mathcal{N}$ be a subcategory of $\operatorname{mod}(KI)$ and $m \in \mathbb{N}$ let us denote by $\mu^1_{\mathcal{N}}(m)$ the less possible cardinality of an almost parameterizing family for $\operatorname{ind}_m(\mathcal{N})$. Given a vector $v \in \mathbb{N}^I$ denote by $\overline{\mu}^1_{\mathcal{N}}(v)$ the less possible cardinality of an almost parameterizing family for the category of indecomposable objects in $\mathcal{N}$ with coordinate vector equal to v.

Following [16, Section 14.4] we say that $\mathcal{N}$ is of *linear* (resp. *polynomial*) *growth* if there exists an integer g such that $\mu^1_{\mathcal{N}}(t) \leq gt$ for any $t \geq 1$ (resp. $\mu^1_{\mathcal{N}}(t) \leq g^t$ for any $t \geq 2$).

The poset I is said to be of *tame prinjective type* (resp. *finite prinjective type*) if $\operatorname{prin}(KI)$ is of tame representation type (resp. finite representation type). Similarly, we say that I is of *linear* (resp. *polynomial*) growth if so is $\operatorname{prin}(KI)$.

It follows from [18, Proposition 2.4] that $\operatorname{prin}(KI)$ is of tame representation type (resp. linear, polynomial growth) if and only if I-spr is of tame representation type (resp. linear, polynomial growth).

The following lemma gives a description of tameness and the growth of $\operatorname{prin}(KI)$ and I-spr in terms of coordinate vectors.

Lemma 2. *Assume that $\mathcal{N}$ is one of the categories* $\operatorname{prin}(KI)$, I-spr.

(a) *The category $\mathcal{N}$ is of tame representation type if and only if for any $v \in \mathbb{N}^{n+m}$ there exists an almost parameterizing family for the category $\operatorname{ind}_{(v)}(\mathcal{N})$ formed by indecomposable objects of $\mathcal{N}$ having the coordinate vector equal to v.*

(b) *The category $\mathcal{N}$ is of linear (resp. polynomial) growth if and only if there exists an integer g such that $\overline{\mu}^1_{\mathcal{N}}(v) \leq t|v|$ for any $v \in \mathbb{N}^I$ (resp. $\overline{\mu}^1_{\mathcal{N}}(v) \leq |v|^g$ for any $v \in \mathbb{N}^I$ such that $|v| \geq 2$).*

Proof. Apply the arguments used in the proof of [18, Proposition 2.4]. □

Let us finish this section with discussion of the relations between the representation type of a poset and of its peak subposet. Assume that I is an n-peak poset, $n \geq 1$, and L is a peak subposet of I. Following [16, Section 17.5] let us define the commutative diagram of categories and functors

$$\begin{array}{ccc} \mathrm{mod_{sp}}(KL) & \overset{T_L}{\underset{\mathrm{res}_L}{\rightleftarrows}} & \mathrm{mod_{sp}}(KI) \\ \simeq\big\downarrow \Theta'_L & & \simeq\big\downarrow \Theta'_I \\ L\text{-spr} & \overset{T'_L}{\underset{\mathrm{res}'_L}{\rightleftarrows}} & I\text{-spr} \end{array} \tag{2.4}$$

by the formulas $T_L(X) = X \otimes_{KL} (e_L K I e_{(I - \cup \max I)})$, $\mathrm{res}_L(Y) = Ye_L$, where $e_J = \sum_{i \in J} e_i$ for any subposet $J \subseteq I$. Note that there is an algebra isomorphism $KL \cong e_L K I e_L$. The functor res'_L is the restriction functor $(M_i)_{i\in I} \mapsto (M_i)_{i \in L}$, and T'_L associates to $\mathbf{M} = (M_s)_{s\in L}$ in L-spr the peak I-space $T'_L(\mathbf{M}) = (M^{\vee}_i)_{i \in I}$, where $M^{\vee}_p = M_p$ for $p \in \max L$, $M^{\vee}_p = 0$ for $p \in \max I \setminus \max L$, and

$$M^{\vee}_j = \sum_{j \succeq s \in L} \pi_j(M_s) \subseteq \bigoplus_{j \preceq p \in \max I} M_p \subseteq M^{\bullet}$$

for $j \in I \setminus \max I$.

Lemma 3.

(a) *The diagram (2.4) is commutative. The functor T_L is full, faithful and right exact, res_L is exact, T_L is left adjoint to res_L and $\mathrm{res}_L \circ T_L \cong \mathrm{id}$.*

(b) *Given a module X in $\mathrm{mod_{sp}}(KL)$ the number $\mathbf{cdn}(T_L(X))(i)$ equals $\mathbf{cdn}(X)(i)$ if $i \in L$ and zero otherwise. Moreover, the image of T_L is the full subcategory of $\mathrm{mod_{sp}}(KI)$ formed by objects Y such that $\mathbf{cdn}(Y)(j) = 0$ for $j \notin L$.*

(c) *If the poset I is of tame prinjective type (resp. of linear, polynomial growth), then L is of tame prinjective type (resp. of linear, polynomial growth).*

Proof. The commutativity of (2.4) follows by a simple analysis. For the proof of (b) and the remaining part of (a) we refer to [16, Theorem 17.46]. In order to prove (c) assume that $v \in \mathbb{N}^L$ and let $\overline{v} \in \mathbb{N}^I$ be defined by $\overline{v}(i) = v(i)$ if $i \in L$ and $\overline{v}(i) = 0$ otherwise. Assume that the functors

$$\widehat{N}^{(l)} = (-) \otimes_{K[t]} N^{(l)} : \mathrm{ind}_1(\mathrm{mod}(K[y])) \to \mathrm{mod}(KI),$$

where $l = 1, \ldots, m$, form an almost parameterizing family for the category $\mathrm{ind}_{(\overline{v})}(I\text{-spr})$ of indecomposable peak I-spaces having the coordinate vector $\overline{v}$. It is clear that the family of functors $\mathrm{res}_L \circ \widehat{N}^{(l)} = (-)\otimes_{K[t]} N^{(l)} e_L$, $l = 1, \dots, r$, is an almost parameterizing family for $\mathrm{ind}_{(v)}(L\text{-spr})$. Thus the statement follows from Lemma 2. □

3. Posets with weakly non-negative Tits form

Apart from the Tits quadratic form $q_I : \mathbb{Q}^I \to \mathbb{Q}$ we also shall use the symmetric bilinear form

$$(-,-)_I : \mathbb{Q}^I \times \mathbb{Q}^I \longrightarrow \mathbb{Q} \tag{3.1}$$

associated to q_I by the formula $(x,y)_I = \frac{1}{2}[q_I(x+y) - q_I(x) - q_I(y)]$.

A basic role in our further considerations is played by the following two theorems.

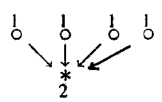
Theorem 4 (see [15,18]). *Suppose that K is an infinite field and I is a finite poset having exactly two maximal elements $\star$ and $+$. Then the following conditions are equivalent:*

(a) *The category I-spr of peak I-spaces is of finite representation type.*

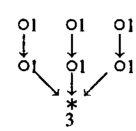
(b) *The rational Tits quadratic form $q_I : \mathbb{Q}^I \to \mathbb{Q}$ (1.1) is weakly positive, that is, $q_I(z) > 0$ for all nonzero vectors $z \in \mathbb{N}^I$.*

(c) *The poset I does not contain as a peak subposet a poset isomorphic to any of the following one-peak enlargements*

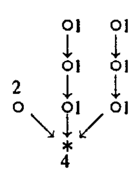
$\mathcal{P}_1 = \mathcal{K}_1^*$:



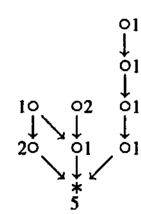
$\mathcal{P}_2 = \mathcal{K}_2^*$:



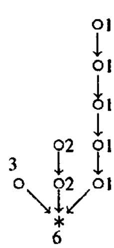
$\mathcal{P}_3 = \mathcal{K}_3^*$:



$\mathcal{P}_4 = \mathcal{K}_4^*$:

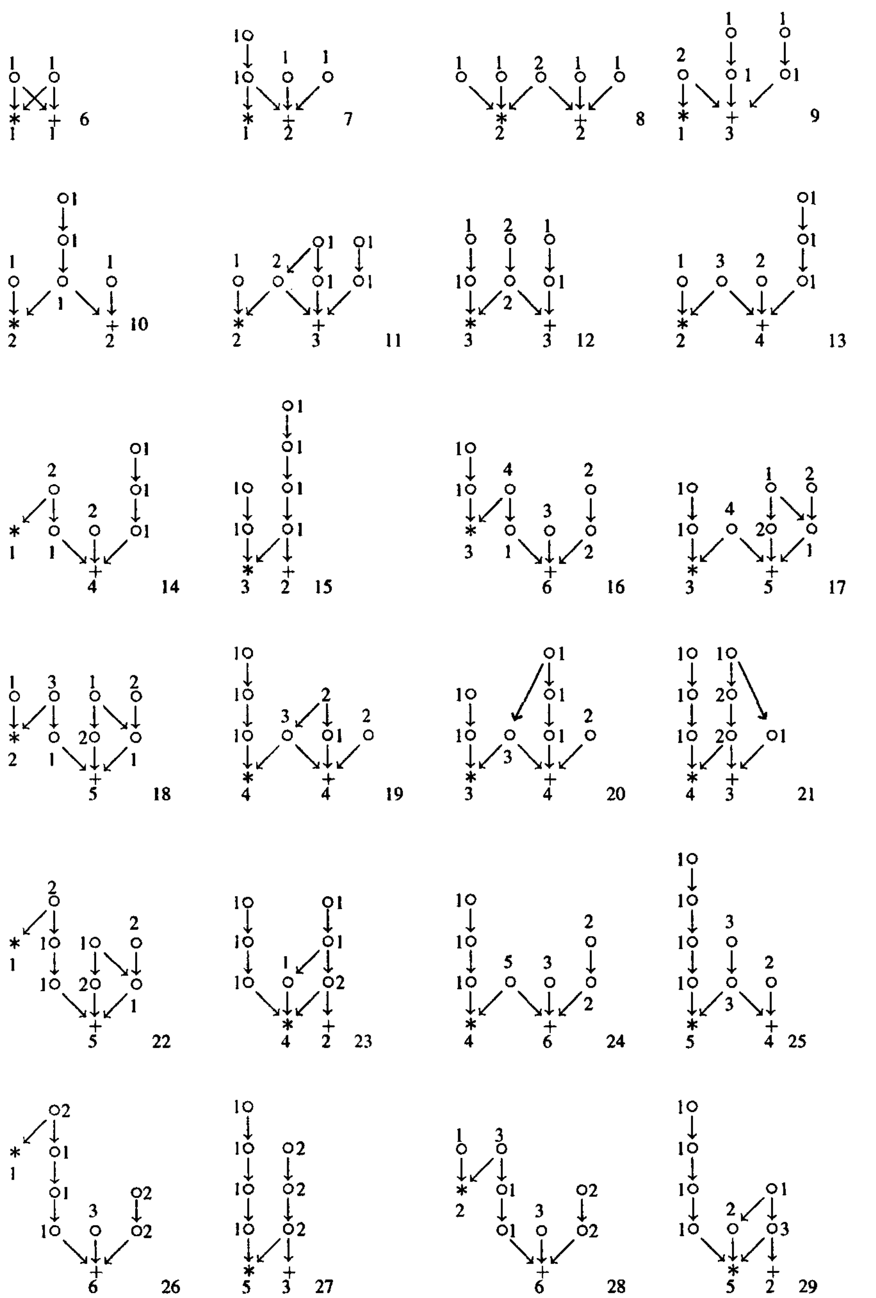


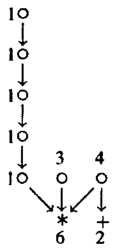
$\mathcal{P}_5 = \mathcal{K}_5^*$:



of the critical posets $\mathcal{K}_1 = (1,1,1,1)$, $\mathcal{K}_2 = (2,2,2)$, $\mathcal{K}_3 = (1,3,3)$, $\mathcal{K}_4 = (N,4)$, $\mathcal{K}_5 = (1,2,5)$ of Kleiner (see [16]) and to any of the 26 two-peak *posets $\mathcal{P}_6, \ldots, \mathcal{P}_{31}$ listed in Table* 2. □

Table 2. *Minimal two-peak posets $\mathcal{P}_6, \ldots, \mathcal{P}_{31}$ of infinite prinjective type.* In the following tables the numbers marked in the diagram $\mathcal{P}_j$ are the coordinates of a vector $\mu_{\mathcal{P}_j} \in \mathbb{N}^{\mathcal{P}_j}$ which generates the group $\operatorname{Ker} q_{\mathcal{P}_j} := \{x \in \mathbb{Z}^{\mathcal{P}_j} \mid q_{\mathcal{P}_j}(x) = 0\} = \mathbb{Z}\mu_{\mathcal{P}_j}$.





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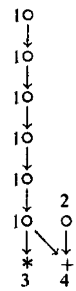

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Theorem 5 (see [8,9]).

(1) *For any two-peak poset I of infinite prinjective type the following conditions are equivalent:*

(a) *The Tits quadratic form $q_I : \mathbb{Q}^I \to \mathbb{Q}$ (1.1) is weakly non-negative, that is, $q_I(z) \geq 0$ for all vectors $z \in \mathbb{N}^I$.*

(b) *I contains as a peak subposet a poset $\mathcal{P} = \mathcal{P}_j$ of one of the types $\mathcal{P}_1, \dots, \mathcal{P}_{31}$ and $(\overline{\boldsymbol{\mu}}_{\mathcal{P}_j}, \varepsilon_a)_I \geq 0$ for all $a \in I$, where ε_a is the ath standard basis vector of $\mathbb{Z}^I$, $\boldsymbol{\mu}_{\mathcal{P}} \in \mathbb{N}^{\mathcal{P}}$ is a generator of the kernel*

$$\operatorname{Ker} q_{\mathcal{P}} := \{v \in \mathbb{Z}^{\mathcal{P}} \mid q_{\mathcal{P}}(v) = 0\}$$

of $q_{\mathcal{P}}$ and $\overline{\boldsymbol{\mu}}_{\mathcal{P}} \in \mathbb{N}^I$ is defined by formula $\overline{\boldsymbol{\mu}}_{\mathcal{P}}(j) = \boldsymbol{\mu}_{\mathcal{P}}(j)$ for $j \in \mathcal{P}$ and $\overline{\boldsymbol{\mu}}_{\mathcal{P}}(j) = 0$ for $j \in I \setminus \mathcal{P}$. The coordinates of the vectors $\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_5$ are listed in Theorem 4, *whereas the vectors $\boldsymbol{\mu}_6, \dots, \boldsymbol{\mu}_{31}$ are listed in* [18, Section 5] *and in Table* 2.

(c) *The poset I does not contain as a peak subposet any of the posets $\mathcal{N}_1^*, \dots, \mathcal{N}_6^*$ and the two-peak posets listed in Table* 1.

(d) *The category* $\operatorname{prin}(KI)$ *is not of fully wild representation type, that is, there is no full and faithful exact functor* $\operatorname{mod}\left(\begin{smallmatrix} K & K^3 \\ 0 & K \end{smallmatrix}\right) \to \operatorname{prin}(KI)$ *(see* [19,9]).

(2) *If the category I*-spr *is of tame representation type then the Tits quadratic form $q_I : \mathbb{Q}^I \to \mathbb{Q}$ is weakly non-negative.* $\square$

An alternative proof of the equivalence (a) $\Leftrightarrow$ (c) follows from the recent results of v. Höhne [6].

In Section 5 the converse of statement (2) in Theorem 5 will be proved under some assumption on the poset I. In this case the wildness of $\operatorname{prin}(KI)$ implies fully wildness (see [9]). Unfortunately we do not know if this implication holds for an arbitrary two-peak poset I (see the "Note added in proof").

4. An upper chain reflection

We recall that a vector space category is a system $\mathbb{K}_K = (\mathbb{K}, |-|_K)$, where $\mathbb{K}$ is a K-category and $|-|_K : \mathbb{K} \to \operatorname{mod}(K)$ is a faithful K-linear functor. A subspace category $\mathcal{U}(\mathbb{K}_K)$ of $\mathbb{K}_K$ has as objects all triples (U, X, ϕ), where U is in $\operatorname{mod}(K)$, X is an object of $\mathbb{K}$ and $\phi : U_K \to |X|_K$ is a K-linear map. A morphism $f : (U, X, \phi) \to (U', X', \phi')$ is a pair $f = (f_0, f_1)$, where $f_0 : U \to U'$ is a K-linear map, $f_1 : X \to X'$ is a morphism in $\mathbb{K}$ and $|f_1| \circ \phi = \phi' \circ f_0$ (see [16, Section 17.1]).

Suppose that $\max I = \{*, +\}$ and the subposet $C = \star^\nabla \cap +^\nabla$ of I is linearly ordered with $\max C = \{c\}$. We set

$$I_* = I \setminus +^\nabla, \qquad I_+ = I \setminus \star^\nabla, \qquad I_C = I \setminus C = I_* \cup I_+, \tag{4.1}$$

where $j^\nabla = \{i \in I \mid i \preceq j\}$ for any $j \in I$. Note that the set $I_* \cap I_+$ is empty and I_*, I_+ are one-peak posets. Consider the vector space category (see [12], [16, Section 17.1])

$$\mathbb{H}_K^C = \mathrm{Hom}_{KI_C}(\mathrm{rad}(e_c KI), I_C\text{-spr}), \tag{4.2}$$

where $\mathrm{rad}(e_c KI)$ is the Jacobson radical of the right ideal $e_c KI$. Suppose in addition that every indecomposable object

$$\overline{y} := \mathrm{Hom}_{KI_C}(\mathrm{rad}(e_c KI), y)$$

in the category $\mathbb{H}_K^C$ is one-dimensional. It follows that the set $I(\mathbb{H}_K^C)$ consisting of the isomorphism classes of indecomposable objects $\overline{y}$ in $\mathbb{H}_K^C$ is a poset with respect to the relation

$$\overline{x} \preceq \overline{y} \Leftrightarrow \mathbb{H}_K^C(\overline{y}, \overline{x}) \neq 0.$$

According to [14, 3.3] we associate with our two-peak poset I the disjoint union

$$\xi_C I = C \cup I(\mathbb{H}_K^C) \tag{4.3}$$

equipped with a partial order extending the partial orders in C and $I(\mathbb{H}_K^C)$ by the following new relations

$$\overline{y} \prec c \quad \text{for all } \overline{y} \in I(\mathbb{H}_K^C), \text{ and}$$
$$i \prec \overline{y} \Leftrightarrow \mathrm{Hom}_{KI_C}(\mathrm{res}(e_i KI), y) = 0 \quad \text{for } i \in C \setminus \{c\},\ \overline{y} \in I(\mathbb{H}_K^C).$$

Here we denote by $\mathrm{res}(e_i KI)$ the restriction of the representation $e_i KI$ of I to the subposet $I_C = I_* \cup I_+$ of I.

We call the one-peak poset $\xi_C I$ the *reflection of I with respect to the chain C*.

The main result of this section is the following theorem.

Theorem 6. *Suppose that I is a two-peak poset such that the subposet $C = \star^\nabla \cap +^\nabla$ of I is linearly ordered. Under the notation above the following hold:*

(i) *If every indecomposable object $\overline{y}$ of the category $\mathbb{H}_K^C$ (4.2) is one-dimensional then*

(i_1) *the set $\xi_C I$ (4.3) associated with I is a one-peak poset and there is an equivalence of categories*

$$I\text{-spr}/[I_*\text{-spr}, I_+\text{-spr}] \cong \xi_C I\text{-spr}/[C\text{-}\overline{\mathrm{spr}}], \tag{4.4}$$

where $I_* = I \setminus +^\nabla$, $I_+ = I \setminus *^\nabla$ and C-$\overline{\text{spr}}$ is the full subcategory of $\xi_C I$-spr consisting of objects $\mathbf{M} = (M_j)_{j \in \xi_C I}$ with $M_j = M_c$, $\{c\} = \max C$, for all $j \in \xi_C I \setminus C$. There is an equivalence of categories C-$\overline{\text{spr}} \cong C$-spr;

(i₂) *the category I-spr is of tame representation type if and only if the category $\xi_C I$-spr is of tame representation type.*

(ii) *Assume that any of the following two conditions is satisfied:*

(ii₁) *the category I-spr is of tame representation type;*

(ii₂) *the Tits quadratic form $q_I : \mathbb{Q}^I \to \mathbb{Q}$ is weakly non-negative and the element c is maximal in the poset $I \setminus \{*, +\}$.*

Then every indecomposable object in $\mathbb{H}_K^C$ is one-dimensional and there exists an equivalence of categories (4.4).

Proof. (i) The first part of (i) follows from [14, Theorem 3.4]. In view of the equivalence of categories I-spr $\cong \text{mod}_{\text{sp}}(KI)$ given in (2.2) the required equivalence (4.4) is induced by the functors presented in the diagram (3.11) in [14]. The second part of (i) may be proved by showing that the functors in the diagram (3.11) in [14] preserve and respect parameterizing families of functors up to a suitable localization of $K[t]$. This can be done by a simple modification of the arguments used in the proof of Lemma 15.39 and Proposition 15.46 in [16] and in [18, Proposition 2.4] (see [7]).

(ii) Assume that the category I-spr is of tame representation type and denote by $\mathcal{S}(\mathbb{H}_K^C)$ the full subcategory of the subspace category $\mathcal{U}(\mathbb{H}_K^C)$ consisting of all objects (U_K, y, t), with the map $t : U_K \to |y|_K$ injective. Consider the two-peak subposet $L = \{c\} \cup I_* \cup I_+$ of I and the full dense functor

$$\Phi : L\text{-spr} \longrightarrow \mathcal{S}(\mathbb{H}_K^C)$$

defined as follows (compare with [12] and [16, 17.19]). Since $c \in L$ is a minimal element, then we can view the algebra KL as a triangular matrix algebra

$$KL = \begin{pmatrix} K & N \\ 0 & S \end{pmatrix},$$

where $S = KI_C = KI_* \times KI_+$ and $N = \text{rad}(e_c KI)$ is a K-S-bimodule. In view of the equivalence L-spr $\cong \text{mod}_{\text{sp}}(KL)$ in (2.2), every peak L-space $\mathbf{M}$ can be viewed as a socle projective KL-module of the form $\mathbf{M} = (M_c, M'', h)$, where M_c is a K-vector space, M'' is the restriction of $\mathbf{M}$ to $I_C = I_* \cup I_+$ (viewed as a socle projective S-module) and $h : M_c \otimes {}_K N_S \to M''_S$ is an S-homomorphism. Note that the map $\overline{h} : M_c \longrightarrow \text{Hom}_S({}_K N_S, M''_S)$ adjoint to h is injective, because $\mathbf{M}$ is a peak L-space. We set $\Phi(\mathbf{M}) = (M_c, \overline{M}'', \overline{h})$, where $\overline{M}'' = \text{Hom}_S(N_S, M''_S)$. Since $\overline{h}$ is injective, $\Phi(\mathbf{M})$ is an object of $\mathcal{S}(\mathbb{H}_K^C)$. The functor Φ is defined on maps in a natural way. By applying the same type of arguments as in [4, Section 5] one can show that Φ preserves tame representation type. Since L is a peak subposet of I then there exists a pair of functors (see (2.4))

$$L\text{-spr} \underset{\text{res}'_L}{\overset{T'_L}{\rightleftarrows}} I\text{-spr}$$

and since by the assumption (ii_1), the category I-spr is of tame representation type, then according to Lemma 3(c) L-spr is also of tame representation type and therefore the category $\mathcal{S}(\mathbb{H}_K^C)$ is of tame representation type.

Assume to the contrary that the category $\mathbb{H}_K^C$ has an indecomposable object $\overline{y}$ of dimension $r > 1$. Without loss of generality we may suppose that y is an indecomposable object in I_C-spr. Since $I_C = I_* \cup I_+$, then $I_C\text{-spr} \cong I_*\text{-spr} \times I_+\text{-spr}$ and therefore y is either in I_*-spr or in I_+-spr. Assume that $y \in I_*$-spr and let $z = e_+KI_+$ be the unique simple projective peak I_+-space. Denote by $\mathbb{K}_K$ the full vector space subcategory of $\mathbb{H}_K^C$ generated by the objects $\overline{y}$ and $\overline{z}$. Then the subspace category $\mathcal{U}(\mathbb{K}_K)$ is a full subcategory of $\mathcal{S}(\mathbb{H}_K^C)$ and $\mathcal{S}(\mathbb{K}_K)$ is equivalent with the category of K-linear representations of the quiver

$$\overline{z} \bullet \longleftarrow \overset{c}{\bullet} \begin{array}{c}\longrightarrow\\ \vdots\\ \longrightarrow\end{array} \bullet \overline{y} \quad \Big\} \quad (r \text{ arrows})$$

having no simple injective summand. Since $r \geq 2$, the quiver is of wild representation type (see [12] and Examples 11–13 in [16, Section 14.3]). Therefore the category $\mathcal{S}(\mathbb{H}_K^C)$ is not of tame representation type [4] and we get a contradiction which finishes the proof in case (ii_1) is satisfied.

For the proof of (ii) in case the condition (ii_2) is satisfied assume that $\mathbb{H}_K^C$ has an indecomposable object $\overline{y}$ with $\dim_K \overline{y} \geq 2$. Note that $\mathrm{rad}(e_cKI) = e_*KI \oplus e_+KI$ and given an object y in $I_C\text{-spr} \cong I_*\text{-spr} \times I_+\text{-spr}$ we have $\dim_K \overline{y} = \dim_K y(e_* + e_+)$. Without loss of generality we can assume that $\overline{y} = \mathrm{Hom}_{KI_C}(\mathrm{rad}(e_cKI), y)$ and y is an indecomposable peak I_*-space.

Since I_* is a one peak poset then according to [16, Theorem 11.52] there exists a unique preprojective component $\mathcal{P}(I_*)$ in I_*-spr.

First we shall show that there exists an indecomposable peak I_*-space y in $\mathcal{P}(I_*)$ such that $\dim_K \overline{y} \geq 2$.

Assume to the contrary that $\dim_K \overline{y} = 1$ for any indecomposable object y in $\mathcal{P}(I_*)$. It follows that $\mathrm{soc}(y)$ is simple and therefore y is a submodule of the injective envelope of the unique simple KI_*-module. By Lemma 5.12 in [16] the component $\mathcal{P}(I_*)$ is finite. It follows from Corollary 11.54 in [16] that $\mathcal{P}(I_*) = I_*$-spr and therefore $\dim_K \overline{y} = 1$ for any indecomposable object y in I_*-spr, contrary to our assumption.

Let y be an indecomposable object y in $\mathcal{P}(I_*)$ such that $\dim_K \overline{y} \geq 2$. It follows from [16, Theorem 11.52] that $\mathrm{End}(y) \cong K$.

Denote by z the simple projective KI-module e_+KI and by $S(c)$ the simple KI-module $\mathrm{top}(e_cKI) \cong e_cKI/\mathrm{soc}(e_cKI)$. Note that $S(c)$ is prinjective, $\mathrm{End}(S(c)) \cong K$, $\mathrm{End}(z) \cong K$.

According to Lemma 11.38 in [16] there exists a KI-module epimorphism $y^+ \to y$, where y^+ is an indecomposable prinjective KI-module such that $\Theta_I(y^+) \cong y$. Since c is maximal in $I \setminus \{*, +\}$ the KI-module $\mathrm{rad}(e_cKI)$ is projective and therefore we get

$$\dim_K \mathrm{Hom}_{KI}(\mathrm{rad}(e_cKI), y^+) \geq \dim_K \mathrm{Hom}_{KI}(\mathrm{rad}(e_cKI), y) = \dim_K \overline{y} \geq 2.$$

It is easy to see that $\mathrm{Hom}_{KI}(e_cKI, y^+) = 0$, $\mathrm{Hom}_{KI}(e_cKI, z) = 0$ and $\mathrm{Hom}_{KI}(\mathrm{rad}(e_cKI), z) \cong K$. Then using the exact sequence $0 \to e_*KI \oplus e_+KI \to e_cKI \to S(c) \to 0$ we show that $\dim_K \mathrm{Ext}^1_{KI}(S(c), y^+) \geq 2$ and $\dim_K \mathrm{Ext}^1_{KI}(S(c), z) = 1$.

It is easy to see that $\mathrm{Hom}_{KI}(S(c), y^+) = 0$, $\mathrm{Hom}_{KI}(S(c), z) = 0$, $\mathrm{Hom}_{KI}(y^+, z) = 0$ and $\mathrm{Hom}_{KI}(z, y^+) = 0$. Since $(\mathbf{cdn}\, y^+)(c) = (\mathbf{cdn}\, z)(c) = 0$, we get $\mathrm{Hom}_{KI}(y^+, S(c)) = \mathrm{Hom}_{KI}(z, S(c)) = 0$ (see the proof of Lemma 3.4(c) in [9]). Since y is preprojective then according to Theorem 11.80 in [16] the module y^+ is preprojective in $\mathrm{prin}(KI_*)$ and $\mathrm{End}(y^+) \cong K$.

It follows that for the prinjective KI-module $U = S(c)^2 \oplus (y^+)^2 \oplus z$ we have $\dim_K \mathrm{End}(U) = 9$ and $\dim_K \mathrm{Ext}^1_{KI}(U, U) \geq 10$. Then [11, Proposition 4.4] yields

$$q_I(\mathbf{cdn}\, U) = \dim_K \mathrm{End}(U) - \dim_K \mathrm{Ext}^1_{KI}(U, U) \leq -1$$

which contradicts the assumption (ii_2).

Hence any of the conditions (ii_1), (ii_2) implies that every indecomposable object of $\mathbb{H}^C_K$ is one-dimensional. This completes the proof of the theorem. □

Corollary 7. *Let I be a finite two-peak poset with* $\max I = \{*, +\}$ *and let K be an algebraically closed field. Suppose that I does not contain as a peak subposet the poset*

$$\widetilde{\mathbb{A}}_3^* : \begin{tikzcd} \bullet \arrow[d] \arrow[dr] & \bullet \arrow[d] \arrow[dl] \\ \star & + \end{tikzcd}$$

or equivalently, the subposet $C := *^{\nabla} \cap +^{\nabla}$ of I is linearly ordered.

*Then the category I-*spr *is of tame representation type if and only if every indecomposable object of the category $\mathbb{H}^C_K$ (4.2) is one-dimensional and the category $\xi_C I$-*spr *is of tame representation type, or equivalently, the one-peak poset $\xi_C I$ (4.3) does not contain as a peak subposet the one-peak enlargements $\mathcal{N}_1^*, \dots, \mathcal{N}_6^*$ of Nazarova's posets listed in Theorem 1.*

Proof. Apply [14, Theorem 3.4], Theorem 6 and Nazarova's theorem [16, Theorem 15.3]. □

The description of the poset $\xi_C I$ given in (4.3) is not very constructive, because the formula (4.3) depends essentially on the Auslander–Reiten quiver of the category I_C-spr $\cong I_*$-spr $\times$ I_+-spr and on the position of $\mathrm{rad}(e_cKI)$ in the quiver. The poset $\xi_C I$ can be described explicitly in terms of I if the poset I is upper chain reducible in the sense of Definition 8 below. In this case we shall also obtain a useful formula (4.6) relating the Tits quadratic forms q_I and $q_{\xi_C I}$, and we shall prove that the weakly non-negativity of q_I implies the tameness of I-spr.

Definition 8. A poset I is defined to be *upper chain reducible* if $\max I = \{*, +\}$ and the following conditions are satisfied.

(i) The subposet

$$C = *^{\nabla} \cap +^{\nabla} = \{c_1 \prec c_2 \prec \cdots \prec c_m = c\}$$

of I is linearly ordered.

(i′) The element $c \in C$ is maximal in $I \setminus \{*, +\}$.

(ii) The subposets $I_* = I \setminus +^{\nabla}$ and $I_+ = I \setminus *^{\nabla}$ of I do not contain a triple of pairwise incomparable elements.

The following result is a consequence of [14, Corollary 3.13].

Corollary 9. *If I is an upper chain reducible poset and* max $I = \{*, +\}$ *then $\mathbb{H}_K^C$ has only finitely many isomorphism classes of indecomposable objects, each indecomposable object of $\mathbb{H}_K^C$ is one-dimensional and the one-peak poset $\xi_C I$ has the form*

$$\xi_C I = C \cup \widehat{I}_* \cup \widehat{I}_+ \quad (\textit{disjoint union}),$$

where $\widehat{I}_p = (I_p \setminus \{p\}) \cup \{\overline{p}\} \cup \{s \vee t \mid s, t \text{ are incomparable in } I_p\}$ for any $p \in \{*, +\}$, $\overline{p}$ is a new element associated with p and $s \vee t = \{s, t\}$. The partial order relation in $\xi_C I$ is an extension of the partial order $\preceq$ in $C \cup (I_* \setminus \{*\}) \cup (I_+ \setminus \{+\})$ by the following new relations:

(a) $\overline{p} \prec u$ *for all* $u \in \widehat{I}_p$,

(b) $u \preceq s \vee t \Leftrightarrow (u \preceq s$ *or* $u \preceq t)$,

(c) $s \vee t \preceq u \Leftrightarrow (s \preceq u$ *and* $t \preceq u)$,

(d) $s \vee t \preceq s' \vee t' \Leftrightarrow$ *each of s, t is less or equal to one of s', t'.*

(e) $x \preceq c$ *for all* $x \in \xi_C I$. □

In order to obtain the connection (4.6) below between the Tits quadratic forms q_I and $q_{\xi_C I}$ we define the linear map

$$\widetilde{\xi} : \mathbb{Q}^{\xi_C I} \to \mathbb{Q}^I \tag{4.5}$$

by the formula

$$\begin{aligned}
\widetilde{\xi}(\eta'_{\overline{p}}) &= -\eta_p \quad \text{for } p \in \{*, +\}, \\
\widetilde{\xi}(\eta'_u) &= \eta_u \quad \text{for } u \in I_C - \{*, +\}, \\
\widetilde{\xi}(\eta'_c) &= \eta_c + \eta_* + \eta_+, \\
\widetilde{\xi}(\eta'_{s \vee t}) &= \eta_s + \eta_t + \eta_p \quad \text{for } s, t \in I_p, \\
\widetilde{\xi}(\eta'_{c_j}) &= \eta_{c_j} - \eta_c \quad \text{for } j \le m - 1,
\end{aligned}$$

where $\{\eta'_j\}$, $\{\eta_i\}$ are the standard bases of $\mathbb{Q}^{\xi_C I}$ and of $\mathbb{Q}^I$ respectively.

As an immediate consequence of the definition we obtain the following corollary.

Corollary 10. *If I is an upper chain reducible poset*, max $I = \{*, +\}$, $C = *^{\nabla} \cap +^{\nabla} = \{c_1 \prec c_2 \prec \cdots \prec c_m = c\}$, $y' \in \mathbb{Q}^{\xi_C I}$ *and* $y = \widetilde{\xi}(y') \in \mathbb{Q}^I$, *then*

$$y_p = y'_c - y'_{\overline{p}} + \sum_{s\vee t\in \widehat{I_p}} y'_{s\vee t} \quad \textit{for } p \in \max I = \{*,+\},$$

$$y_{c_j} = y'_{c_j} \quad \textit{for } j = 1,\dots,m-1,$$

$$y_c = y'_c - (y'_{c_1} + \cdots + y'_{c_{m-1}}),$$

$$y_j = y'_j + \sum_{j\vee s\in \widehat{I_p}} y'_{j\vee s} \quad \textit{for } j \in I_p,\ p \in \max I = \{*,+\},$$

where I_p is the set defined in (4.1).

Proposition 11. *Suppose that I is a two-peak upper chain reducible poset with* $\max I = \{*,+\}$, *and let $\widetilde{\xi} : \mathbb{Q}^{\xi_c I} \to \mathbb{Q}^I$ be the linear map* (4.5).

(a) *The defect quadratic form*

$$q_I^- = q_{\xi_c I} - q_I \circ \widetilde{\xi} : \mathbb{Q}^{\xi_c I} \to \mathbb{Q}^I \tag{4.6}$$

has the form

$$q_I^-(y') = \sum_{\beta\prec\gamma\in\xi_c I} b_{\beta\gamma} y'_\beta y'_\gamma$$

for any $y' \in \mathbb{Q}^{\xi_c I}$, where

$$b_{c_j\gamma} = -1 \quad \textit{if } \gamma = s\vee t \textit{ and } c_j \prec s,\ c_j \prec t,\ j \le m-1,$$

$$b_{\overline{p}\gamma} = 1 \quad \textit{if } \overline{p} \prec \gamma = s\vee t,$$

$$b_{\beta u} = 1 \quad \textit{if } \beta = s\vee t \textit{ and either } u \prec s \textit{ or } u \prec t,$$

$$b_{r\vee u, s\vee t} = \begin{cases} 1 & \textit{if } s,t,r,u \textit{ form the poset } \begin{smallmatrix} r & & u \\ \downarrow & & \downarrow \\ s & & t \end{smallmatrix} \textit{ in } I_p, \\ -1 & \textit{if } s,t,r,u \textit{ form the poset } \begin{smallmatrix} r & & u \\ \downarrow & \times & \downarrow \\ s & & t \end{smallmatrix} \textit{ in } I_p, \end{cases}$$

$p \in \max I = \{*,+\}$ *and I_p is the set defined in* (4.1). *The remaining coefficients $b_{\beta\gamma}$ are zero.*

(b) *The poset I does not contain peak subposets of one of the forms*

 (4.7)

if and only if $b_{\beta\gamma} \ge 0$ for all $\beta \preceq \gamma$ in the poset $\xi_c I$.

Proof. (a) Let A, A', B be the symmetric matrices of the quadratic forms q_I, $q_{\xi_c I}$ and $\sum_{\beta\prec\gamma\in\xi_c I} b_{\beta\gamma} y_\beta y_\gamma$, respectively. If D is the matrix of $\widetilde{\xi}$ in the standard bases, then $D^{\mathrm{tr}}AD$ is the matrix of $q_I \circ \widetilde{\xi}$ and it is sufficient to check that $A' - D^{\mathrm{tr}}AD = B$, or equivalently, that $\eta'_\beta(A' - D^{\mathrm{tr}}AD)(\eta'_\gamma)^{\mathrm{tr}} = \eta'_\beta B(\eta'_\gamma)^{\mathrm{tr}}$ for all $\beta, \gamma \in \xi_c I$. We leave it to the reader.

Statement (b) follows from (a). □

5. Proof of Theorem 1

Before we prove the main result of our paper let us state the following lemma.

Lemma 12. *Suppose that I is an upper chain reducible poset and* $\max I = \{*, +\}$. *If I does not contain as a peak subposet a poset isomorphic to one of the one-peak posets $\mathcal{N}_1^*, \dots, \mathcal{N}_6^*$, or to one of the two-peak posets presented in Table* 1, *then the one-peak poset $\xi_C I$* (4.3) *does not contain as a peak subposet any of the posets $\mathcal{N}_1^*, \dots, \mathcal{N}_6^*$ up to isomorphism.*

Proof. By our assumption and Corollary 9 the set $\xi_C I$ (4.3) is a poset, and $\xi_C I$ has the form described in Corollary 9. Throughout we shall use the notation introduced there.

The lemma will be proved by applying the linear map $\widetilde{\xi} : \mathbb{Q}^{\xi_C I} \to \mathbb{Q}$ (4.5) and an analysis of the defect quadratic form $q_I^- : \mathbb{Q}^{\xi_C I} \to \mathbb{Q}$ (4.6) defined by the formula

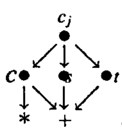
$$q_I^-(y') = q_{\xi_C I}(y') - q_I(\widetilde{\xi}(y')) = \sum_{\beta \prec \gamma \in \xi_C I} b_{\beta\gamma} y'_\beta y'_\gamma,$$

where the coefficients $b_{\beta\gamma} \in \mathbb{Z}$ are as in Proposition 11.

Let us start the proof by the following useful property of the quadratic form q_I^-, which easily follows from Proposition 11.

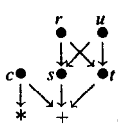
(A) The coefficient $b_{\beta\gamma}$ of q_I^- is negative if and only if up to a permutation of the peaks $*, +$ one of the following two conditions is satisfied:

(A_1) $\beta = c_j$ for some $j \leq m-1$, $\gamma = svt \in \widehat{I}_+$ and I contains as a peak subposet the poset



.

(A_2) $\beta = rvu \in \widehat{I}_+$, $\gamma = svt \in \widehat{I}_+$ and I contains as a peak subposet the poset



.

The following observation will be also essential in the proof of the lemma.

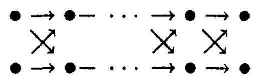
(B) If the posets I_* and I_+ does not contain the subposet $(\bullet\ \bullet\ \bullet)$ consisting of three incomparable elements and (A_1) or (A_2) holds, then the following three conditions are satisfied.

(B_1) If (A_1) holds and an element $\alpha \in \xi_C I$ is incomparable with c_j and with γ, then $\alpha \in \widehat{I}_*$.

(B_2) If (A_2) holds and an element $\alpha \in \xi_C I$ is incomparable with β and with γ, then $\alpha \in \widehat{I}_* \cup C$.

(B_3) If $\widehat{I}_*$ contains a subposet of one of the types $(\bullet\ \bullet\ \bullet)$, $(\bullet\ \begin{smallmatrix}\bullet\\ \downarrow\\ \bullet\end{smallmatrix})$, then the poset I_* contains a subposet of type $(\bullet\ \begin{smallmatrix}\bullet\\ \downarrow\\ \bullet\end{smallmatrix})$ and I contains as a peak subposet a poset of type $\widehat{\mathcal{P}}_8$.

The proof of (B_1) and (B_2) follows from a simple analysis of the order relation $\preceq$ in $\xi_C I$. For the proof of (B_3) we note that if I_* does not contain subposets of type $(\bullet\ \begin{smallmatrix}\bullet\\ \downarrow\\ \bullet\end{smallmatrix})$, then I_* is a subposet of the garland

$\mathcal{G}$: 

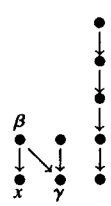
and it follows from the definition of $\xi_C I$ that $\widehat{I}_*$ is also a subposet of a garland $\mathcal{G}$, contrary to our assumption. Note also that if I_* contains $(\bullet\ \begin{smallmatrix}\bullet\\ \downarrow\\ \bullet\end{smallmatrix})$ then $(\bullet\ \begin{smallmatrix}\bullet\\ \downarrow\\ \bullet\end{smallmatrix}) \cup \{c, s, t, *, +\}$ is a subposet of I of type $\widehat{\mathcal{P}}_8$.

In order to prove the lemma assume to the contrary that the poset $\xi_C I$ contains as a peak subposet a poset $\mathcal{N}^* = \mathcal{N} \cup \{*\}$ of one of the types $\mathcal{N}_1^*, \dots, \mathcal{N}_6^*$. Consider two cases.

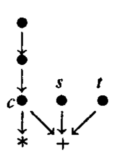
Case 1: $\mathcal{N}$ contains two elements $\beta \prec \gamma$ such that $b_{\beta\gamma} < 0$. It follows that $\mathcal{N} \in \{\mathcal{N}_2, \dots, \mathcal{N}_6\}$ and from the definition of $\xi_C I$ it follows that either $\beta, \gamma \in \widehat{I}_* \cup C$ or $\beta, \gamma \in \widehat{I}_+ \cup C$. Assume that $\beta, \gamma \in \widehat{I}_+ \cup C$. By the observation above we infer that either (A_1) or (A_2) holds.

First we assume that $\mathcal{N} = \mathcal{N}_2 = (\bullet\ \bullet\ \bullet\ \begin{smallmatrix}\bullet\\ \downarrow\\ \bullet\end{smallmatrix})$. It follows from ($B_1$) and ($B_2$) that in both cases ($A_1$) and ($A_2$) the subposet $(\bullet\ \bullet\ \bullet)$ consisting of the first three elements of $\mathcal{N}_2$ is contained in $\widehat{I}_* \cup C$. If $(\bullet\ \bullet\ \bullet) \subseteq \widehat{I}_*$, then according to ($B_3$) the poset I_* contains a subposet of type $(\bullet\ \begin{smallmatrix}\bullet\\ \downarrow\\ \bullet\end{smallmatrix})$ and it is easy to see that I contains a two-peak subposet of type $\widehat{\mathcal{P}}_8$; contrary to our assumption. If C contains an element of $(\bullet\ \bullet\ \bullet)$, then I contains $\widehat{\mathcal{P}}_7^!$ and again we get a contradiction.

Next assume that $\mathcal{N}$ is of type $\mathcal{N}_5$ and $\mathcal{N}$ has the form

(*) 

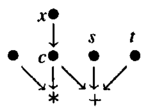
Denote the right-hand chain by L. Since the elements of L are incomparable with β and γ then (B_1) and (B_2) yield $L \subseteq \widehat{I}_* \cup C$ and $|L \cap C| \leq 1$, because if $|L \cap C| > 1$ then the poset



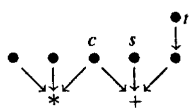
of type $\widehat{\mathcal{P}}_7^3$ is contained in I, which contradicts our assumption.

Assume that $L \cap C = \emptyset$. Hence $L \subseteq \widehat{I}_*$ and since I does not contain a subposet of type $\widehat{\mathcal{P}}_8$ then according to (B_3), I_* is a subposet of a garland $\mathcal{G}$ and therefore either I_* contains $(\bullet, \bullet)$ or I_* is linearly ordered and $L \subseteq I_*$. It follows that $x \notin \widehat{I}_*$, because otherwise I_* contains a subposet of type $\left(\bullet\ \begin{smallmatrix}\bullet\\ \downarrow\\ \bullet\end{smallmatrix}\right)$ and I_* cannot be embedded into a garland $\mathcal{G}$.

If $x \in C$, then I contains a peak subposet

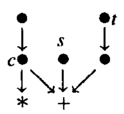


of type $\widehat{\mathcal{P}}_7^1$. If $x \in \widehat{I}_+$, then the assumptions that x, γ are incomparable, $\gamma = s \vee t$ and I_+ contains no three incomparable elements imply that I_+ contains a subposet $\left(\begin{smallmatrix}s\\ \bullet\end{smallmatrix}\ \begin{smallmatrix}\bullet t\\ \downarrow\\ \bullet\end{smallmatrix}\right)$ of type $\left(\bullet\ \begin{smallmatrix}\bullet\\ \downarrow\\ \bullet\end{smallmatrix}\right)$ and therefore I contains a peak subposet



of type $\widehat{\mathcal{P}}_8$ if $(\bullet, \bullet) \subseteq I_*$; or I contains a peak subposet of type $\widehat{\mathcal{P}}_{24}^1$ (formed by L, c, $\left(\begin{smallmatrix}s\\ \bullet\end{smallmatrix}\ \begin{smallmatrix}\bullet t\\ \downarrow\\ \bullet\end{smallmatrix}\right)$, $*$, $+$), if I_* is a chain and $L \subseteq I_*$. In all cases we get a contradiction.

Now we assume that $|L \cap C| = 1$. It follows that we are in the situation of (A_2) and therefore $L' = L \cap \widehat{I}_*$ consists of four elements. Since x is incomparable with all elements of L, then $x \notin C$. If $x \in \widehat{I}_*$, then similarly as in the case $L \cap C = \emptyset$ we conclude that I_* contains $\left(\bullet\ \begin{smallmatrix}\bullet\\ \downarrow\\ \bullet\end{smallmatrix}\right)$ and the poset I contains $\widehat{\mathcal{P}}_8$; a contradiction. If $x \in \widehat{I}_+$, a simple analysis shows that I_+ contains a subposet of type $\left(\begin{smallmatrix}s\\ \bullet\end{smallmatrix}\ \begin{smallmatrix}\bullet t\\ \downarrow\\ \bullet\end{smallmatrix}\right)$, because we are in the situation described in (A_2). It follows that I contains a peak subposet



of type $\widehat{\mathcal{P}}_7^2$ and we get a contradiction.

Now we suppose that $\beta \prec \gamma$ holds in $\mathcal{N}$ and $\mathcal{N} \in \{\mathcal{N}_3, \mathcal{N}_4, \mathcal{N}_6\}$ or $\mathcal{N} = \mathcal{N}_5$ but the relation $\beta \prec \gamma$ in $\mathcal{N}$ is not that one shown in $(*)$. It follows that $\mathcal{N}$ contains a subposet L of type $\left(\bullet\ \begin{smallmatrix}\bullet\\ \downarrow\\ \bullet\end{smallmatrix}\right)$ consisting of elements being incomparable with β and with γ.

From (A_1) and (A_2) we infer that $L \subseteq \widehat{I}_* \cup C$ and since I does not contain a subposet of type $\widehat{\mathcal{P}}_8$, then according to (B_3) the poset L is not contained in $\widehat{I}_*$. It follows that either the chain in L is contained in C, or the upper element of the chain in L is contained in C, or the chain is contained in $\widehat{I}_*$. In these cases I contains as a peak subposet a poset of type $\widehat{\mathcal{P}}_7^3$, $\widehat{\mathcal{P}}_7^1$ and $\widehat{\mathcal{P}}_7^1$, respectively. In all cases we get a contradiction which proves Lemma 12 in Case 1.

Case 2: The coefficients $b_{\beta\gamma}$ of $q_I^-(y')$ are non-negative for all $\beta, \gamma \in \mathcal{N}$.

It follows that for any vector $v \in \mathbb{N}^{\mathcal{N}^*}$ we have $q_I(\widetilde{\xi}(\overline{v})) \le q_{\xi_C I}(\overline{v})$, where $\overline{v}(i) = v(i)$ for $i \in \mathcal{N}^*$ and $\overline{v}(i) = 0$ for $i \in \xi_C I \setminus \mathcal{N}^*$.

If $\mathcal{N}^* = \mathcal{N}_j^*$, $j \in \{1, \dots, 6\}$, then $q_{\mathcal{N}^*}(\mathbf{v}^{(j)}) = -1$, where $\mathbf{v}^{(j)} \in \mathbb{N}^{\mathcal{N}_j^*}$ and $\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(6)}$ are the following vectors:

$\mathbf{v}^{(1)}$	$\mathbf{v}^{(2)}$	$\mathbf{v}^{(3)}$	$\mathbf{v}^{(4)}$	$\mathbf{v}^{(5)}$	$\mathbf{v}^{(6)}$
					2
				2	2
			2	2	2
		2	2 2	2	2
	1	2 2 1	2 1	2 4 1	4 1
1 1 1 1 1	2 2 2 1	2 2 1	4 2 1	4 2 1	6 4 1
2	4	6	8	10	12

respectively. We recall that c is the unique maximal element of $\xi_C I$ and of $\mathcal{N}_j^*$.

If $y' = \mathbf{v}^{(j)}$, then it is easy to see that $y'_c \ge y'_t$ for all $t \in \mathcal{N}_j^*$ and $y'_c \ge \sum_{t \in L} y'_t$ for any linearly ordered subposet L of $\mathcal{N}_j$.

It follows from Corollary 10 that the vector $y = \widetilde{\xi}(\overline{y}') \in \mathbb{Z}^I$ has non-negative coordinates and in view of the inequality

$$q_I(y) \le q_{\xi_C I}(\overline{y}') = q_{\mathcal{N}_j^*}(\mathbf{v}^{(j)}) = -1$$

we infer that q_I is not weakly non-negative. Hence, in view of Theorem 5, the poset I contains as a peak subposet one of the posets $\mathcal{N}_1^*, \dots, \mathcal{N}_6^*$ or one of the two-peak posets in Table 1. This contradiction finishes the proof of Lemma 12. □

Proof of Theorem 1. The implication (a) $\Rightarrow$ (b) follows from Theorem 5(2) and (b) $\Rightarrow$ (c) from Theorem 5(1). In order to prove (c) $\Rightarrow$ (a) assume that I does not contain as a peak subposet any of the Nazarova's posets $\mathcal{N}_1^*, \dots, \mathcal{N}_6^*$ nor the posets presented in Table 1. It follows from Lemma 12 that $\xi_C I$ does not contain as a subposet any of the posets $\mathcal{N}_1^*, \dots, \mathcal{N}_6^*$ and according to Nazarova's Theorem (see [16, Theorem 15.3]) the category $\xi_C I$-spr is of tame representation type. By Theorem 6(i) the category I-spr is of tame representation type. This completes the proof. □

Corollary 13. *If I is an upper chain reducible two-peak poset then the following conditions are equivalent.*

(i) *The category I*-spr *is of tame representation type.*

(ii) *The category* prin(KI) *is of tame representation type.*

(iii) *The category* prin(KI) *is not of wild representation type (see* [16,8]).

(iv) *The category* prin(KI) *is not of fully wild representation type.*

(v) *The Tits quadratic form q_I* (1.1) *of I is weakly non-negative.*

Proof. Apply Theorem 1, [18, Proposition 2.4], [8, Theorem 5.5] and the main result in [9]. □

We recall from [19] that the category $\mathrm{prin}(KI)$ is said to be of fully wild representation type if there exists a full and faithful exact functor $\mathrm{mod}\begin{pmatrix} K & K^3 \\ 0 & K \end{pmatrix} \longrightarrow \mathrm{prin}(KI)$.

Remark 14. (a) We are still not able to prove Theorem 1 for arbitrary two-peak poset I (see the "Note added in proof").

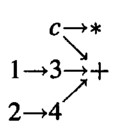
(b) In view of Theorem 6, the equivalence (4.4) provides us a tool for a study of the representation-tame categories I-spr by means of the well-developed representation theory of representation-tame one-peak posets $\xi_C I$ (see [12] and [16]). We hope that this is a way for proving Theorem 1 in general.

(c) We hope that Corollary 13 remains valid for arbitrary n-peak poset I with $n \geq 1$ (see [8] and [9]).

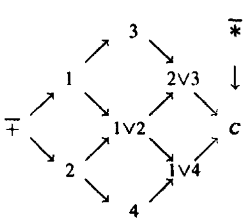
Remark 15. The map (4.5) together with Proposition 11 is very useful in finding vectors in $\mathrm{Ker}\, q_I$. In particular it gives a procedure for determining the vectors $\boldsymbol{\mu}_7, \ldots, \boldsymbol{\mu}_{31}$ in the Table 2 corresponding to those among the posets $\mathcal{P}_7, \ldots, \mathcal{P}_{31}$ presented in Table 2 which are upper chain reducible in the sense of Definition 8. The procedure is based on the kernel vectors $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_5$ (see Thoerems 4(c), 5(b) and [16, (15.19)]) of the critical one-peak posets $\mathcal{P}_1, \ldots, \mathcal{P}_5$ of Kleiner and works as follows.

Suppose that $\mathcal{P} = \mathcal{P}_r$ is one of the posets $\mathcal{P}_7, \ldots, \mathcal{P}_{31}$ which is upper chain reducible. Then $\xi_C \mathcal{P}$ is representation-infinite one-peak poset and c is the unique maximal element of $\xi_C \mathcal{P}$. By theorem of Kleiner the poset $\xi_C \mathcal{P}$ contains as a peak subposet a poset $\mathcal{P}_j \in \{\mathcal{P}_1, \ldots, \mathcal{P}_5\}$. The vector $y' = \overline{\boldsymbol{\mu}}_j \in \mathbb{Q}^{\xi_C \mathcal{P}}$ belongs to $\mathrm{Ker}\, q_{\xi_C \mathcal{P}}$ and it follows from Proposition 11 that $q_I^-(y') = 0$, because a case by case inspection shows that $b_{\beta\gamma} y'_\beta y'_\gamma = 0$ for all $\beta \prec \gamma$ in $\xi_C \mathcal{P}$. It follows that $q_I(\widetilde{\xi}(y')) = q_{\xi_C I}(y') - q^-(y') = 0$ and applying Corollary 10 to $y = \widetilde{\xi}(y') = \widetilde{\xi}(\overline{\boldsymbol{\mu}}_j)$ we calculate that y is the vector $\boldsymbol{\mu}_r$ in the Table 2 corresponding to $\mathcal{P} = \mathcal{P}_r$.

For an illustration consider the poset

$\mathcal{P}$: 

of type $\mathcal{P}_9$. By Corollary 4.7, $\xi_C \mathcal{P}$ has the form



and the elements $\bar{*}$, 3, $1\vee 2$, 4, c form a one-peak subposet of $\xi_C\mathcal{P}$ of type $\mathcal{P}_1$. By [16, (15.19)] the vector $y' = \overline{\boldsymbol{\mu}}_1$ has $y'_c = 2$, $y'_{\bar{*}} = y'_3 = y'_4 = y'_{1\vee 2} = 1$ and $y'_j = 0$ for the remaining indices j. It follows from Proposition 11 that $b_{\beta\gamma}y'_\beta y'_\gamma = 0$ for all $\beta \preceq \gamma$ and therefore $y = \tilde{\xi}(y') \in \operatorname{Ker} q_{\mathcal{P}}$. Applying Corollary 10 to $I = \mathcal{P} = \mathcal{P}_9$ we get $y = \boldsymbol{\mu}_9$.

If $\mathcal{P} = \mathcal{P}_{21}$, then by Corollary 9 $\xi_C\mathcal{P} = \mathcal{P}_4$. Since $\operatorname{Ker} q_{\mathcal{P}_4} = \mathbb{Z}\boldsymbol{\mu}_4$ (see [16, (15.19)]), then applying Corollary 4.9 to $I = \mathcal{P}_{21}$ and $y' = \boldsymbol{\mu}_4$ we get $\tilde{\xi}(\boldsymbol{\mu}_4) = \boldsymbol{\mu}_{21}$.

Note added in proof. Contrary our hope in Remark 14, Theorem 1 does not remain valid for arbitrary poset I. The first author has recently found an example of a two-peak poset which is of wild prinjective type and the Tits form q_I is weakly non-negative.

References

[1] D.M. Arnold, Representations of partially ordered sets and abelian groups, Contemporary Math. 87 (1989) 91–109.
[2] M. Auslander and S.O. Smalø, Almost split sequences in subcategories, J. Algebra 69 (1981) 426–454.
[3] C. Böttinger and R. Göbel, Endomorphism algebras of modules with distinguished partially ordered submodules over commutative rings, J. Pure Appl. Algebra 76 (1991) 211–141.
[4] Ju.A. Drozd, Tame and wild matrix problems, in: Representations and Quadratic Forms, Kiev (1979) 39–74 (in Russian).
[5] P. Gabriel, Indecomposable representations I, Manuscripta Math. 6 (1972) 71–103.
[6] H.J. v. Höhne, On weakly non-negative unit forms and tame algebras, Proc. London Math. Soc., to appear.
[7] S. Kasjan, Adjustment functors and tame representation type, Comm. Algebra 22 (1994) 5587–5597.
[8] S. Kasjan and D. Simson, Varieties of poset representations and minimal posets of wild prinjective type, in: Proceedings of the Sixth International Conference on Representations of Algebras, Canadian Mathematical Society Conference Proceedings, Vol. 14 (1993) 245–284.
[9] S. Kasjan and D. Simson, Fully wild prinjective type of posets and their Tits form, J. Algebra 172 (1995) 506–529.
[10] H. Lenzing and D. Simson, Cohen-Macaulay modules, the completion functor, representations of posets and rational tiled orders, in preparation.
[11] J.A. de la Peña and D. Simson, Prinjective modules, reflection functors, quadratic forms and Auslander-Reiten sequences, Trans. Amer. Math. Soc. 329 (1992) 733–753.
[12] C.M. Ringel, Tame Algebras and Integral Quadratic Forms, Lecture Notes in Mathematics, Vol. 1099 (Springer, Berlin, 1984).
[13] D. Simson, Representations of bounded stratified posets, coverings and socle projective modules, in: Topics in Algebra, Part 1: Rings and Representations of Algebras, Banach Center Publications, Vol. 26 (PWN, Warszawa, 1990) 499–533.
[14] D. Simson, Peak reductions and waist reflection functors, Fund. Math. 137 (1991) 115–144.
[15] D. Simson, Two-peak posets of finite prinjective type, in: Proceedings of the Tsukuba International Conference on Representations of Finite Dimensional Algebras, Canadian Mathematical Society Conference Proceedings, Vol. 11 (1991) 287–298.
[16] D. Simson, Linear Representations of Partially Ordered Sets and Vector Space Categories, in: Algebra, Logic and Applications, Vol. 4 (Gordon & Breach, London, 1992).
[17] D. Simson, Right peak algebras of two-separate stratified posets, their Galois coverings and socle projective modules, Comm. Algebra 20 (1992) 3541–3591.
[18] D. Simson, Posets of finite prinjective type and a class of orders, J. Pure Appl. Algebra 90 (1993) 77–103.
[19] D. Simson, On representation types of module subcategories and orders, Bull. Polish Acad. Sci. Math. 41 (1993) 77–93.
[20] D. Simson, A reduction functor, tameness and Tits form for a class of orders, J. Algebra 174 (1995) 430–452.